SAT Algorithms

6 November, 2020 11:06

- $2^n \cdot \mathrm{poly}(m)$ ... alg for SAT ... trivial
- $2^{n(1-\frac{c}{k})}$ ... alg for k-SAT ... the best known see below

Warm-up.

Thm: Satisfiability of 3-CNF with m clauses can be decided in time $\left(\frac{3}{2}\right)^m \cdot \mathrm{poly}(m)$ with probability of correctness $\geq 1 - \frac{1}{2^n}$.
(randomized algorithm)

Pf:

alg: on input $\psi = C_1 \& C_2 \dots \& C_m$
From each clause with 3 literals, remove one at random.
$\Rightarrow \psi' \dots$ 2-CNF
check whether ψ' is satisfiable,
if it is report ψ is SAT
o/w report ψ is UNSAT.

- 2-SAT can be decided in polynomial time.
- if ψ' is satisfiable $\Rightarrow \psi$ is satisfiable
(... literals from clauses makes ...)

the fle handed to [illegible]

- if ψ is satisfiable $\Rightarrow$ with probability at least $\left(\frac{2}{3}\right)^m$, ψ' is satisfiable

<u>Pf:</u> Let a^* be the sat. assignment of ψ. Each clause is satisfied by making true at least one literal. Probability of removing the satisfying literal $\leq \frac{1}{3}$. Hence, probability we remove literal from a clause and the clause is still satisfied $\geq \frac{2}{3}$. As we do this independently for m clauses the claim follows. $\square$

Hence, if ψ is satisfiable w.p. $\geq \left(\frac{2}{3}\right)^m$ we get a correct answer & if ψ is unsatisfiable, we get always the correct answer.

$\Rightarrow$ repeat the algorithm $\left(\frac{3}{2}\right)^m \cdot n$ - times and report SAT if any of the runs reports SAT

$$\Pr[\text{the output is incorrect}] = \Pr[\text{never reporting SAT for SAT file}] \leq \left(1 - \left(\frac{2}{3}\right)^m\right)^{\left(\frac{3}{2}\right)^m \cdot n} \leq e^{-\left(\frac{2}{3}\right)^m \cdot \left(\frac{3}{2}\right)^m \cdot n} = e^{-n} \leq \frac{1}{2^n}$$

$$= e^{-n} \le \frac{1}{2^n}$$

Fact: $\forall x,\ (1-x) \le e^{-x}$

$\boxed{}$

Thm: There is an alg. for k-SAT running in time $2^{n\left(1-\frac{c}{k\cdot 2^k}\right)} \cdot \mathrm{poly}(m)$ for some $c>0$.

Pf:

alg: on input $\psi = C_1 \wedge C_2 \dots C_m$

pick the shortest clause $C_i = (\ell_1 \vee \ell_2 \vee \dots \ell_{k'})$

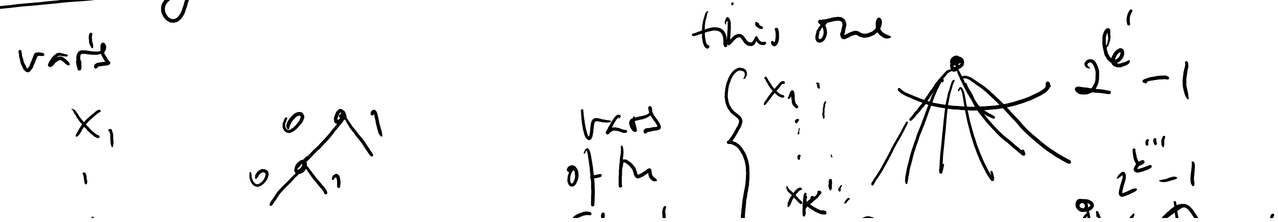
try all possible $2^{k'}-1$ assignments which satisfy C_i, substitute for them into $\psi \to \psi'$ and call recursively on the ψ'

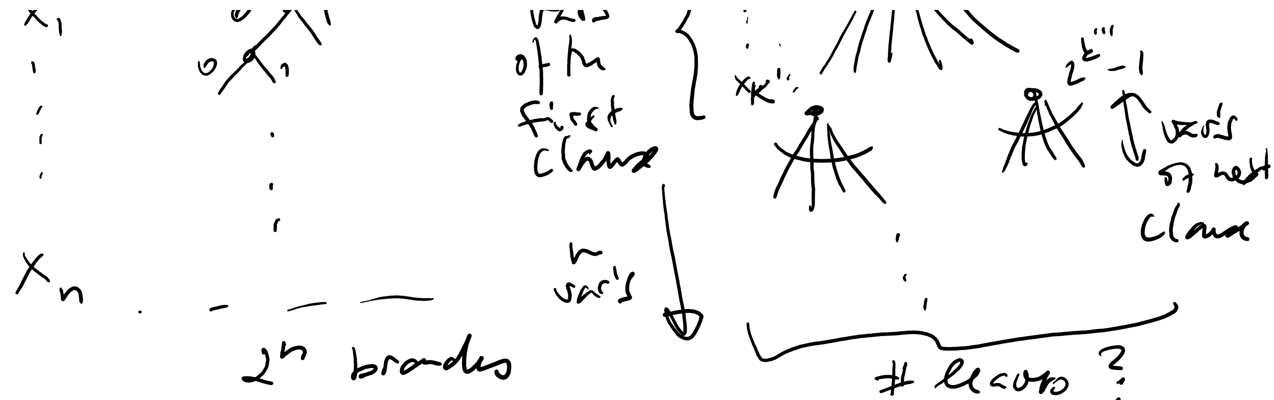
If ψ has no clause ~~...~~ report SAT

If ψ has some empty clause return UNSAT

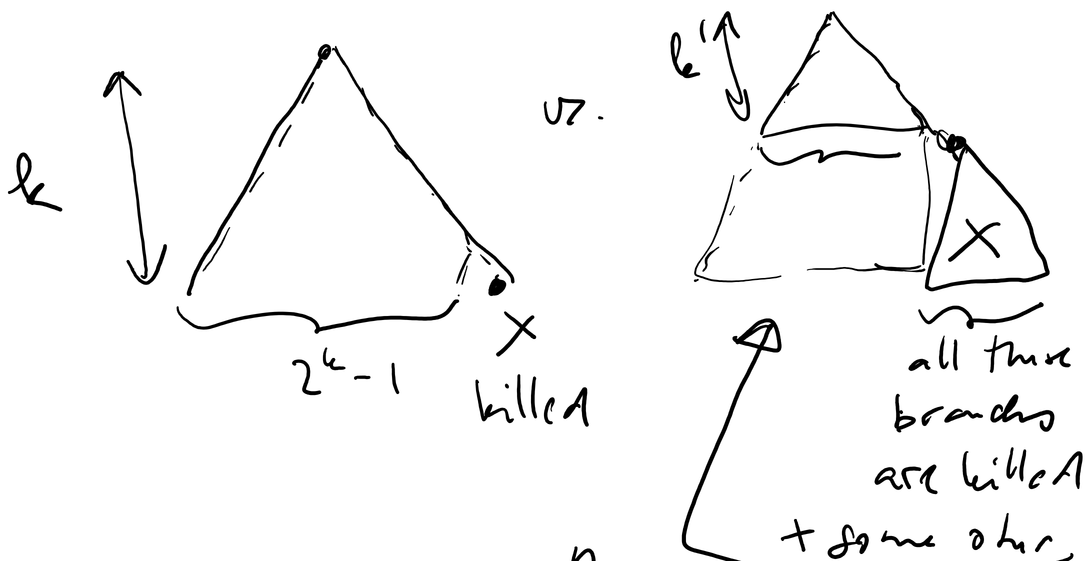
remove from clauses falsified literals and remove clauses containing satisfied literal

Naive alg:





the worst case is we get always k-clause
as the relative savings for k-clauses
is smaller than for k'-clause, $k' < k$:



$$\text{size of the tree} \leq (2^k - 1)^{\frac{n}{k}} = 2^n \left(1 - \frac{1}{2^k}\right)^{n/k}$$

$$= 2^n \cdot e^{-\frac{1}{2^k} \cdot \frac{n}{k}}$$

$$= 2^n \cdot 2^{\frac{\lg_2 e}{k \cdot 2^k}}$$

$$= 2^{n\left(1 - \frac{\lg e}{k\, 2^k}\right)}$$

$\square$

ohlnut:

$n\left(1 - \frac{c}{2^k}\right)$

$2^{n(1-\frac{c}{2k})}$ alg.

alg: on ψ

If ψ has no clause report SAT. If ψ contains empty clause return UNSAT.

Pick a shortest clause $\ell_1 \vee \ell_2 \dots \vee \ell_{k'}$

Recurse on: $\psi|_{\ell_1=1}$, $\psi|_{\ell_1=0, \ell_2=1}$, ...

and $\psi|_{\ell_1=0, \ell_2=0, \dots \ell_{k'-1}=0, \ell_{k'}=1}$

This prunes the tree even more $\boxed{B}$

$\longrightarrow 2^{n(1-\frac{c}{2k})}$ branches

- Randomized alg. for 2-SAT

alg: on input ψ .. 2-CNF

$\psi = C_1 \& C_2 \dots \& C_m$

set $a = 0^n$

repeat $10n^2$-times:

if C_i is falsified by $\underline{a}$, pick one of the variables in C_i at random and flip its value $\to \underline{a}$

If $\underline{a}$ satisfies ψ report SAT

report UNSAT.

$\Rightarrow$ Correct on unsat. f-le

Let a^* be a fixed sat. assignment to ψ

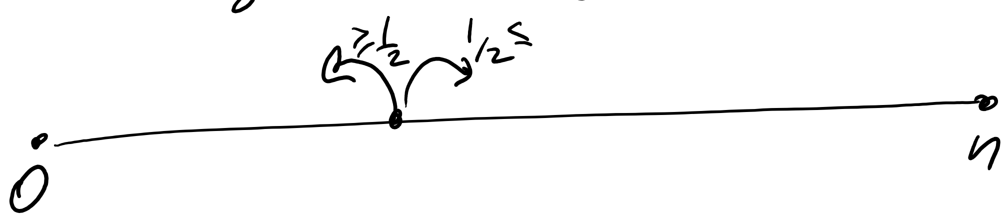
if $\psi(a) = \text{False}$ & C_i is falsified by $\underline{a}$

$\Rightarrow$ a^* & a differ on at least one variable in C_i. With prob. $\geq \frac{1}{2}$ we pick that variable and flip it

$\Rightarrow$ $\mathrm{Ham}(a^*, a)$ decreases w.p. $\geq \frac{1}{2}$

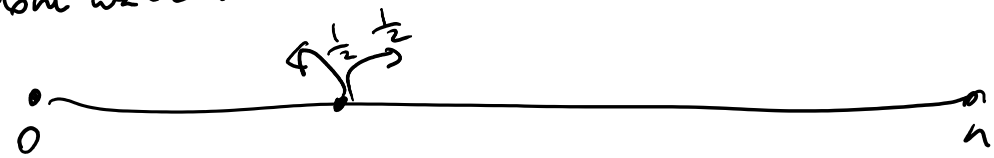
↳ Hamming distance

it might increase by 1 if we are unlucky



$\mathrm{Ham}(a^*, a)$

random walk on line



<u>Fact</u>: The expected time to reach 0 from n (or any other vertex) $\leq n^2$.

$\Pr[$we don't reach a satisfying assignment (e.g. a^*) within $10n^2$ steps$] \leq \frac{1}{10}$

Markov ineq.

- The 2-SAT alg. is correct w.p. ≥ 0.9.

→ <u>Alg. for 3-SAT</u> (Schöning)

<u>alg:</u> on input ψ ... 3-CNF

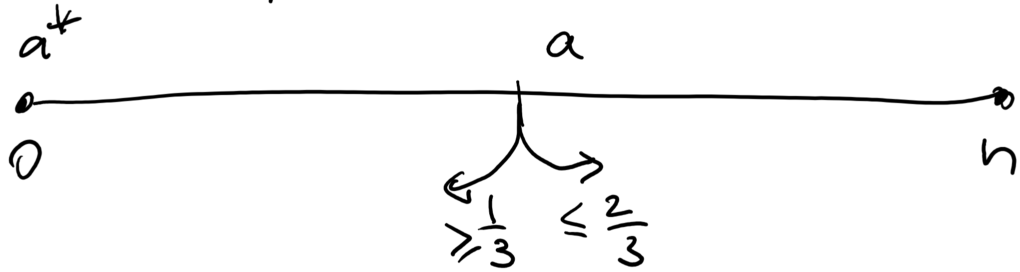
pick <u>a</u> at random from $\{0,1\}^n$

repeat $\frac{n}{2}$-times:

if a clause C_i of ψ is not satisfied by a, pick one of its var's at random and flip its assignment → a.

If $\psi(a)$=true report SAT

report UNSAT.



W/ probability $\frac{1}{2}$, $\mathrm{Ham}(a^*, a) \leq \frac{n}{2}$.

initial one

$\Pr[$in each step we pick the right var. to flip$] \geq \frac{1}{3}$.

$\Rightarrow \Pr[$The alg. reaches $a^*] \geq \frac{1}{2} \cdot \left(\frac{1}{3}\right)^{n/2}$.

$\wedge|$

$\Pr[$alg. is correct on sat fle$]$

<u>repeat</u> $\geq n \cdot 3^{n/2}$ -times $\Rightarrow \Pr[$alg is correct$] \geq 1 - 2^{-n}$

repeat the whole alg. $2n \cdot 3^{n/2}$ - times $\Rightarrow$ Pr [alg is correct] $\geq 1 - 2^{-n}$

$\rightarrow 1.733^n$ algorithm

Improvement: repeat the inner loop $3n$ times.

$$\Pr[\text{error}] \leq \left(\frac{3}{4}\right)^n \cdot \frac{1}{3n} \rightarrow \left(\frac{4}{3}\right)^n = 1.33..^n \text{ alg.}$$

Claim: $\Pr[\text{the inner loop reaches } a^* \text{ from } a]$ (a initial)

$$\geq \left(\frac{1}{2}\right)^{Ham(a^*, a)} \cdot \frac{1}{3n}$$

$$\Pr[\text{error}] \leq \frac{1}{2^n} \sum_{a \in \{0,1\}^n} \Pr[\text{the inner loop reaches } a^* \text{ from } a]$$

$$\leq \frac{1}{2^n} \sum_{t=0}^{n} \binom{n}{t} \frac{1}{2^t} \cdot \frac{1}{3n} = \frac{1}{2^n} \cdot \frac{1}{3n} \cdot \sum \binom{n}{t} \frac{1}{2^t}$$

$$= \frac{1}{2^n} \cdot \frac{1}{3n} \cdot \left(1 + \frac{1}{2}\right)^n = \left(\frac{3}{4}\right)^n \frac{1}{3n}$$

binomial theorem on

Pf: If the algorithm makes $2t$ steps towards a^*

$t = Ham(a^*, a)$ and t steps away then it ends up in a^*.

(consider only the first $3t$ steps out of $3n$ steps)

$\Pr[\text{making } 2t \text{ steps to } a^* \text{ \& } t \text{ away from } a^*]$

$$\geq \binom{3t}{t} \cdot \left(\frac{1}{3}\right)^{2t} \cdot \left(\frac{2}{3}\right)^{t}$$

$H(x) = x\lg\frac{1}{x} + (1-x)\lg\frac{1}{1-x}$

$$\geq \frac{2^{H(\frac{1}{3})3t}}{3t} \cdot \left(\frac{1}{3}\right)^{2t} \cdot \left(\frac{2}{3}\right)^{t}$$

Fact:

$$\binom{s}{\alpha s} \geq \frac{2^{H(\alpha)s}}{s}$$

$$\geq \frac{3^{\frac{1}{3}\cdot 3t} \cdot \left(\frac{3}{2}\right)^{\frac{2}{3}\cdot 3t}}{3t} \cdot \left(\frac{1}{3}\right)^{2t} \left(\frac{2}{3}\right)^{t}$$

$$\geq \frac{3^{t} \cdot \left(\frac{3}{2}\right)^{2t}}{3t} \cdot \left(\frac{1}{3}\right)^{2t} \cdot \left(\frac{2}{3}\right)^{t} = \frac{1}{2^t} \cdot \frac{1}{3t}$$

$1 \leq t \leq n$ $\boxed{}$

Generalization to k-SAT:

alg: input φ ... k-CNF

pick a random $a \in \{0,1\}^n$

repeat $\frac{n}{k}$-times

if $\varphi(a)$ is true report SAT

pick a ~~[illegible]~~ variable at random from some unsatisfied clause & flip its value $\boxed{}$

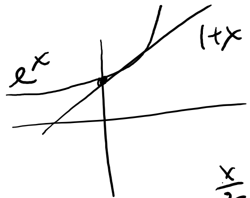
let a^* be a sat. assignment for φ

$$\Pr[\text{ alg. reaches } a^*] \geq \binom{n}{n/k} \cdot \left(\frac{1}{k}\right)^{\frac{n}{k}} \cdot \frac{1}{2^n}$$

initial $\qquad n \qquad n(1-\frac{1}{k})$

initial assignment is at distance $\frac{n}{k}$ from a^*

$$\geq \frac{k^{\frac{n}{k}} \left(\frac{1}{1-\frac{1}{k}}\right)^{n(1-\frac{1}{k})}}{n} \cdot \left(\frac{1}{k}\right)^{\frac{n}{k}} \cdot \frac{1}{2^n}$$



$$\geq \frac{\left(1+\frac{1}{k-1}\right)^{n\left(\frac{k-1}{k}\right)}}{n} \cdot \frac{1}{2^n}$$

$1+x \geq e^{\frac{x}{2}}$

for $x \leq \frac{1}{2}$ $\longrightarrow$ $$\underset{k>2}{\geq} \frac{e^{\frac{1}{2}\cdot\frac{1}{k-1}\cdot\frac{k-1}{k}\cdot n}}{n} \cdot \frac{1}{2^n} = 2^{-n} \cdot e^{\frac{n}{2k}}$$

Hence, the probability of finding a sat. assignment

$$\geq 2^{-n} \cdot 2^{n \cdot \frac{\lg e}{2k}} \cdot \frac{1}{n}$$

Repeating the algorithm $2^{n\left(1 - \frac{\lg e}{2} \cdot \frac{1}{k}\right)} \cdot n^2$ times

gives success probability $\geq 1 - \frac{1}{2^n}$. ∎